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Extreme wave statistics

1 Introduction

OrcaFlex creates realisations of irregular sea states by using a linear combination of linear wave components. The wave components have frequencies and amplitudes that are determined by the frequency spectrum that characterises the sea state. The phases of the wave components are randomised. Separate realisations of a sea-state can be achieved by changing the seed used by the phase random number generator.

This short note compares the statistics of wave time series generated by OrcaFlex with expected theoretical properties, as predicted by the Rayleigh distribution.

2 Empirical data

We consider a sea-state with the following properties:

Spectrum	Torsethaugen
Significant wave height H_s	17.0m
Peak period T_p	19.2s
Mean zero-crossing period T_z	11.3s
Number of wave components	150

Table 1: Sea-state properties

The statistic of interest is the maximum wave elevation in a 3 hour storm duration. We used different seeds to generate 10,000 distinct OrcaFlex realisations of this sea-state, each with 3 hour duration. For each sea-state realisation we then found the maximum value of wave elevation. These 10,000 replicates of 3-hour elevation maxima form our empirical data.

3 Theory

All theory given here is based on Ochi (1981) [1].

By construction, the wave time series in OrcaFlex are Gaussian random processes. For the spectrum considered here the bandwidth parameter ε is 0.87, where ε is defined to be $\epsilon = \sqrt{1 - m_2^2/m_0 m_4}$ and m_i are the spectral moments. Spectra with bandwidth below 0.9 are considered to be *narrow-banded* – it can be seen that our spectrum just meets this criterion.

Extremes of narrow-banded Gaussian processes are known to be well modelled by the Rayleigh probability distribution. Under the Rayleigh assumption the most probable maximum wave elevation for a 3 hour storm duration, Z_{max}, is

$$Z_{max} = \sqrt{2m_0 \ln n}$$

where $n = T/T_z$ is a measure of the number of waves and T is the storm duration.

[1] Ochi, M K, Principles of Extreme Value Statistics and their Application, SNAME Extreme Loads Response Symposium, Arlington, VA, October 19-20 1981.

This equation provides theoretical values for the most probable extreme value. We are also interested in the quantiles of the extreme value distribution, that is the extreme value which is exceeded with a given probability α. Ochi refers to α as the *risk factor* and gives this equation for the corresponding extreme value

$$Z_{max}(\alpha) = \sqrt{2m_0 \ln n/\alpha}. \quad (1)$$

This formula is valid for small values of α, i.e. quantiles in the upper tail of the distribution of maxima.

4 Comparison of empirical and theoretical

We compare the empirical values produced from our 10,000 OrcaFlex realisations against the theoretical values given above. We do this by plotting empirical quantiles together with the corresponding theoretical values.

In more detail we took the 10,000 OrcaFlex realisations of Z_{max} and sorted them in descending order. These ordered realisations are empirical estimates of the quantiles $\{Z_{max}(\alpha_i), i = 1, \ldots, 10{,}000\}$ for risk factors $\alpha_i = i/10{,}000$. Figure 1 compares these empirical estimates with theoretical values given by equation (1).

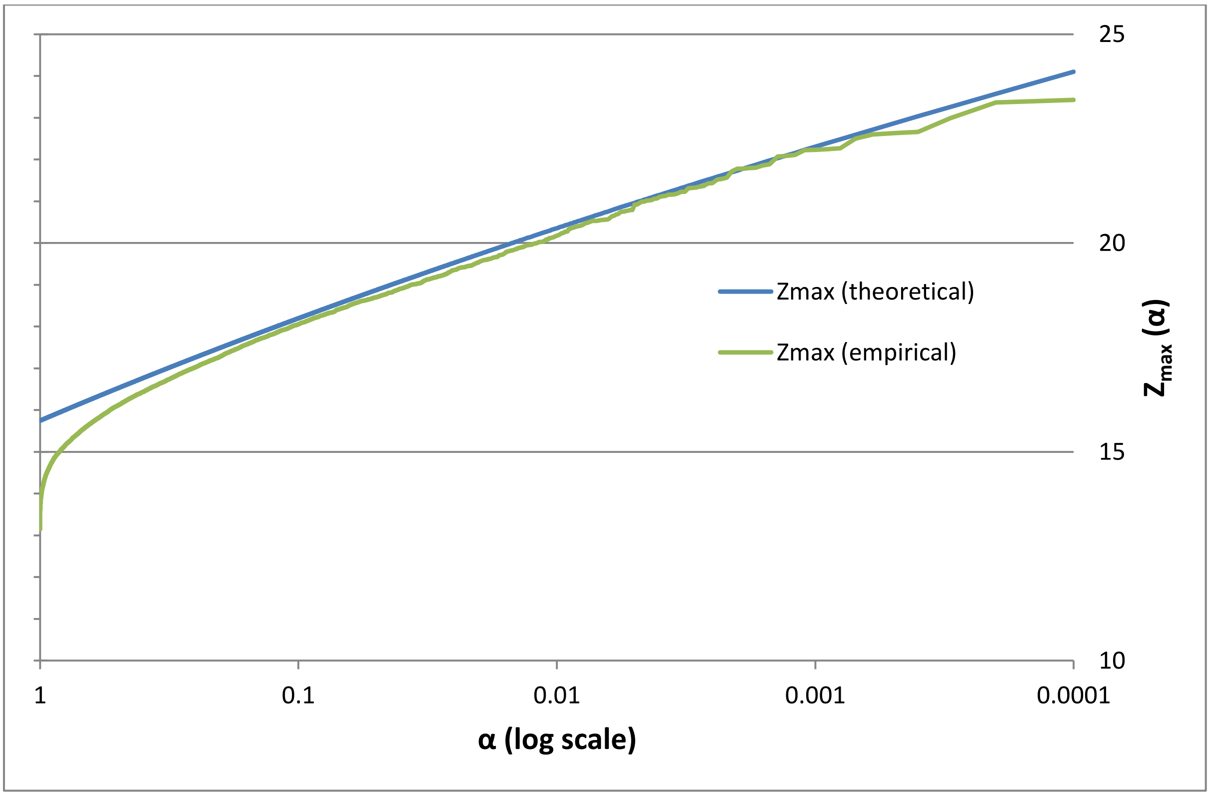

Figure 1: Empirical OrcaFlex quantiles (Torsethaugen spectrum)

First consider the left-hand end of the chart. For values of risk factor α closer to 1 we know that the formula for Z_{max} breaks down, so it is to be expected that the theoretical values deviate from the empirical quantiles.

Over the middle part of the chart the empirical curve follows the theoretical curve very closely.

Towards the right-hand end of the chart (low risk factor) the empirical curve shows more variability. This is to be expected for an empirical quantile estimate, since very few samples from the time series exceed the corresponding extreme value, so the empirical quantile estimate is based on only a small number of samples.

5 Variability of Extreme Elevation of OrcaFlex Random Waves

The empirical curve given above gives the extreme value quantiles of a single realisation of a random process. If we generate data from other realisations of the sea-state then the empirical quantile curve for those new realisations will be different, reflecting statistical variation. We would expect to see more variability towards the right hand end of the chart, since that part of the chart is effectively based on fewer samples, as discussed above.

To illustrate this point, we took our original 10,000 samples of Z_{max} and divided them up into 5 data sets with 2,000 samples in each. This was easier and much faster than generating multiple sets of 10,000 realisations. These were used to produce another plot of quantiles (Figure 2), which shows 5 empirical samples of 2,000 realisations, instead of a single sample of 10,000 realisations. This gives the 5 green empirical extreme value quantile curves in the chart, generated from the OrcaFlex-generated wave time series. As expected, they show statistical variation about the thick blue theoretical quantile curve.

We then also generated 5 empirical extreme value quantile curves by random sampling directly from the Rayleigh distribution, to give an idea of the anticipated statistical variation of the empirical quantiles about the theoretical quantiles. These are shown in the 5 red curves in the chart.

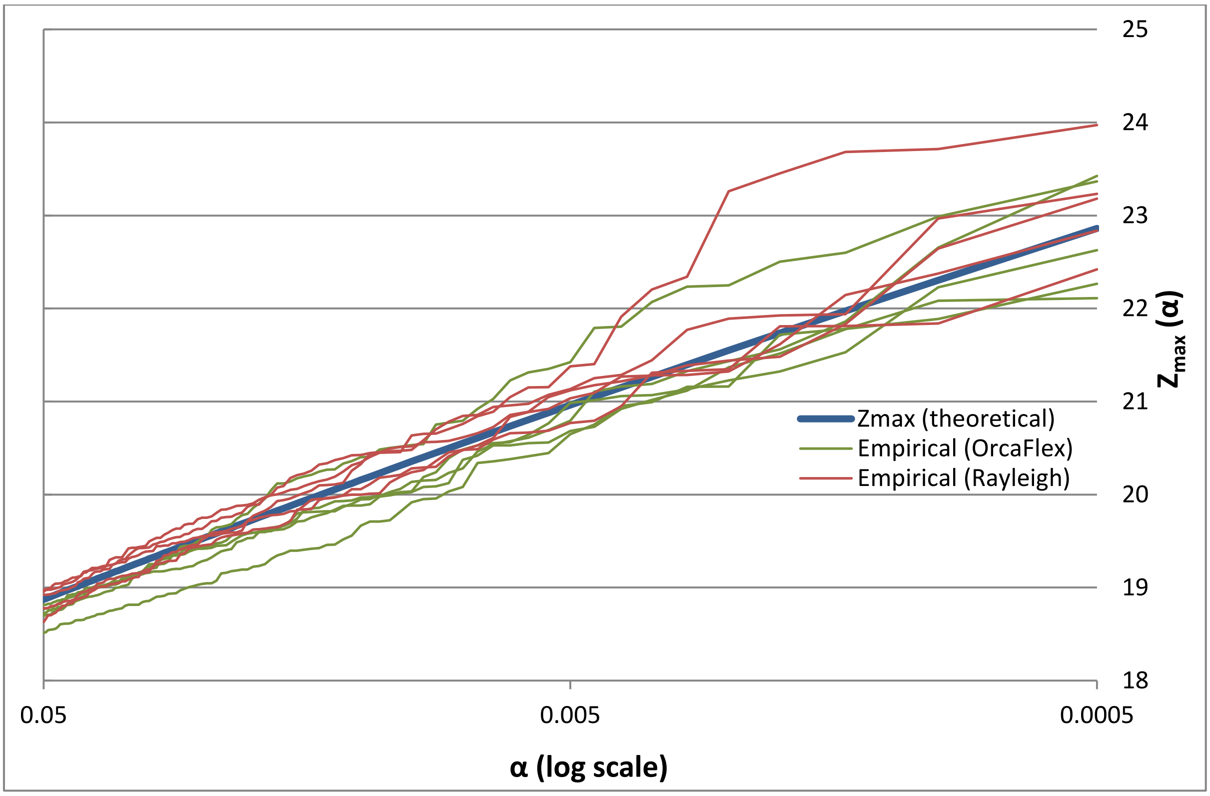

Figure 2: Variability of empirical quantile samples (Torsethaugen spectrum)

The above chart therefore shows:

- A solid blue line showing the theoretical quantile curve.
- 5 green curves showing empirical quantile curves given by 5 samples of 2,000 OrcaFlex realisations each.
- 5 red curves showing empirical quantile curves given by 5 samples from the Rayleigh distribution. These empirical samples from the Rayleigh distribution are included to give an indication of the anticipated statistical variability of the empirical extreme values.

The chart shows that the spread of the empirical samples from OrcaFlex is similar to that of the Rayleigh samples, confirming that the OrcaFlex-generated wave time series have an appropriate level of statistical variation in their extreme values.

With this spectrum we would expect the values from OrcaFlex to be slightly lower than the values from the Rayleigh distribution, because of the relatively high bandwidth of the spectrum used. The curves in Figure 2 do not contradict this expectation.

6 JONSWAP spectrum

We repeated our analysis for a JONSWAP spectrum chosen with the same values of H_s (=17.0m) and T_z (=11.3s) as in the Torsethaugen spectrum analysis. The realisations using this JONSWAP spectrum were generated with 100 wave components, the default value in OrcaFlex.

This JONSWAP spectrum has a bandwidth parameter ε of 0.78 – somewhat smaller than the Torsethaugen spectrum. We would still expect the theoretical values to slightly over-predict extreme values because the theoretical values are based on a perfect narrow-banded assumption.

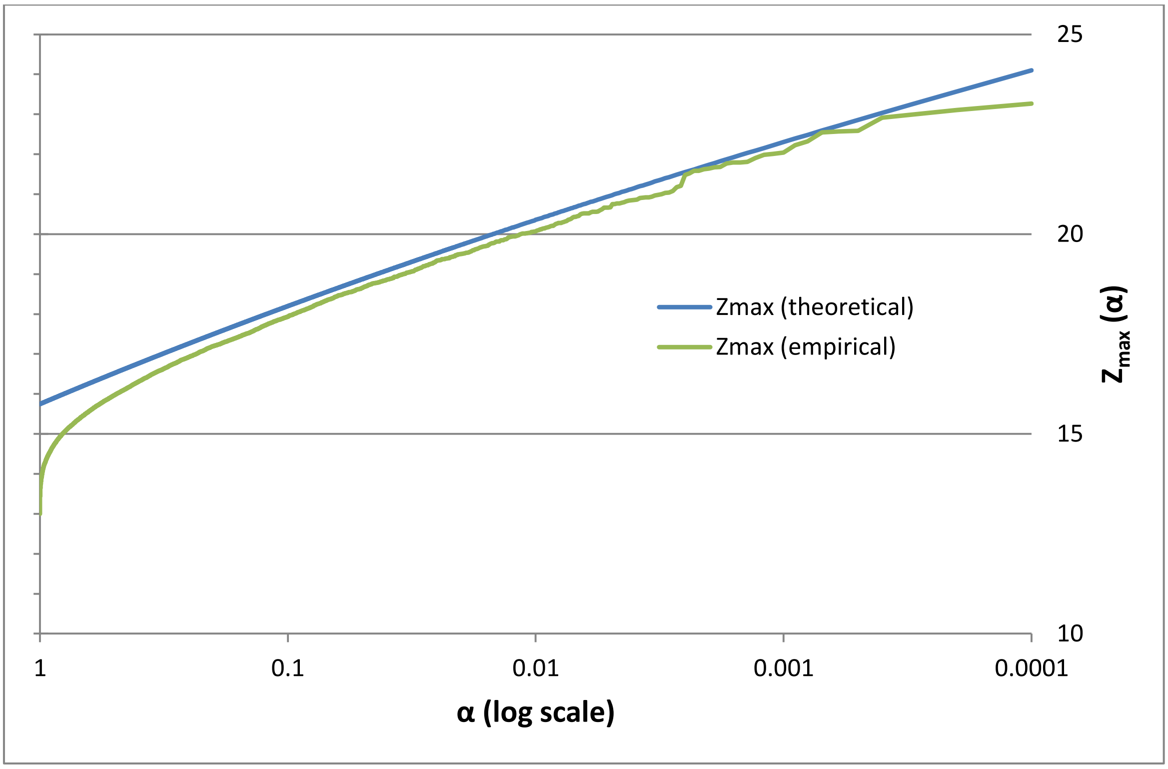

Figure 3: Empirical OrcaFlex quantiles (JONSWAP spectrum)

Figure 3 shows the empirical OrcaFlex quantiles estimated from 10,000 OrcaFlex simulations. The results are very similar to those for the Torsethaugen spectrum shown in Figure 1.

The quantile variability plot for the JONSWAP spectrum (Figure 4) is also very similar to that for the Torsethaugen spectrum (Figure 2). We used a new set of empirical samples from the Rayleigh distribution which is why the red lines in Figure 4 are different from those in Figure 2.

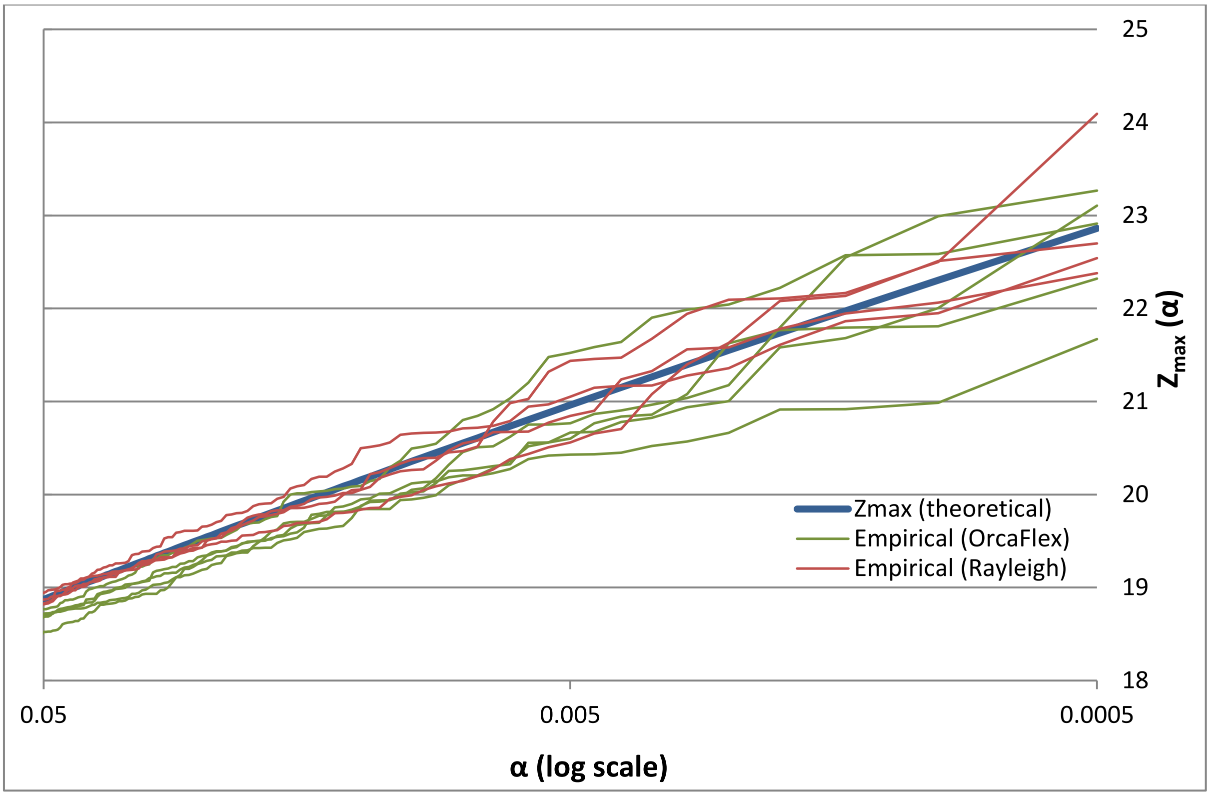

Figure 4: Variability of empirical quantile samples (JONSWAP spectrum)

The results for the JONSWAP spectrum again show nothing to contradict the expectation that the theoretical values slightly over-predict extreme values.

7 Conclusion

The extreme statistics of wave time series produced by OrcaFlex reproduce a number of the properties held by the Rayleigh model.