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# Comparison of OrcaFlex with standard theoretical results

## 1 Introduction

A number of standard theoretical results from literature can be modelled in OrcaFlex. Such cases are, by virtue of being theoretically solvable, quite simple. They do, however, provide a very useful check for the basic mathematical model used by OrcaFlex.

## 2 Analytic catenary equations

### 2.1 Introduction

It is well known that the static equilibrium configuration of an inextensible, flexible line without bend stiffness is a catenary shape.

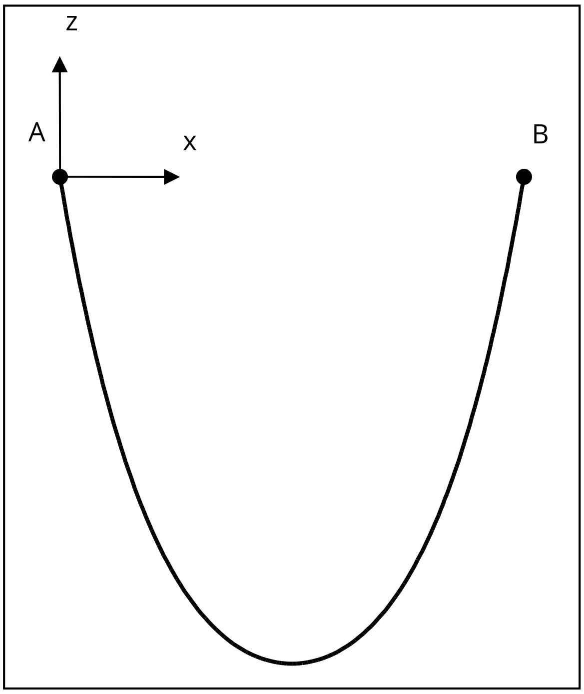


**Figure 1: Catenary shape**

Such a line can be modelled in OrcaFlex and the configuration and tensions predicted by OrcaFlex can be compared with theory. Extremely close agreement is achieved, with differences reducing as segment length reduces.

### 2.2 Catenary theory

The catenary equations are as follows:

$$x = \frac{Th_a}{w}\left[\sinh^{-1}\left(\frac{Tv_a + ws}{Th_a}\right) - \sinh^{-1}\left(\frac{Tv_a}{Th_a}\right)\right]$$

$$z = \frac{Th_a}{w}\left[\sqrt{1+\left(\frac{Tv_a + ws}{Th_a}\right)^2} - \sqrt{1+\left(\frac{Tv_a}{Th_a}\right)^2}\right]$$

where $s$ is the arclength measured from end A, $w$ is the weight per unit length and $Th_a$ and $Tv_a$ are the horizontal and vertical components of tension at end A.

These equations assume that the line is inelastic and so does not stretch axially. However, it is straightforward to modify the equations account for this. If we denote by $K$ the axial stiffness of the line, then the modified catenary equations are:

$$x = \frac{Th_a}{w}\left[\sinh^{-1}\left(\frac{Tv_a + ws}{Th_a}\right) - \sinh^{-1}\left(\frac{Tv_a}{Th_a}\right)\right] + \frac{Th_a s}{K} \tag{1}$$

$$z = \frac{Th_a}{w}\left[\sqrt{1+\left(\frac{Tv_a + ws}{Th_a}\right)^2} - \sqrt{1+\left(\frac{Tv_a}{Th_a}\right)^2}\right] + \frac{Tv_a s}{K} + \frac{ws^2}{2K} \tag{2}$$

### 2.3 Comparison of OrcaFlex with theory

We used these equations to perform a comparison with OrcaFlex using the following line data:

| | |
|---|---|
| Line length $(l)$ | 120 m |
| Weight per unit length $(w)$ | 1.96133 kN/m |
| Axial stiffness $(K)$ | 500 kN |
| Horizontal span | 55 m |
| Vertical span | 0 m |

**Table 1: Catenary data**

We arranged that both ends of the line were at the same vertical position, that is the vertical span is 0. This allows us to calculate directly the vertical component of tension at end A, $Tv_a$.

It is clear that the sum of vertical tension components at the ends, $Tv_a + Tv_b$, equals the total weight of the line, *wl*. Because of symmetry we can see also that $Tv_a$ and $Tv_b$ must be equal, hence $Tv_a = wl/2$.

We now see that there are only two unknowns in equation (1) above, namely $x$ and $Th_a$. If we evaluate the equation at end B, that is for $s = l = 120$, then we can see that $x$ is simply the horizontal span, 55m.

So, we now have an equation with a single unknown, $Th_a$. The equation cannot be rearranged to give a direct expression for $Th_a$ and so we solved it using the goal seek functionality in Microsoft Excel. This results in the theoretical solution $Th_a = 19.87181\text{kN}$.

We modelled the same catenary line in OrcaFlex and produced the following results. We used progressively finer discretisation of the line and observed, as expected, that the difference between the OrcaFlex results and the analytic catenary equations reduced.

| Number of segments | OrcaFlex $Th_a$ (kN) | % difference from theory |
|---|---|---|
| 10 | 19.91354 | 0.2100 |
| 20 | 19.86744 | 0.0220 |
| 30 | 19.86977 | 0.0103 |
| 40 | 19.87066 | 0.0058 |
| 60 | 19.87130 | 0.0026 |
| 120 | 19.87168 | 0.0006 |

**Table 2: Comparison of OrcaFlex with theory**

As a final check we evaluated equation (2) at the mid-point of the line, that is for $s = l/2 = 60$ to find the $z$ coordinate of the catenary at its lowest point. This produced a value of $z = -57.77842\text{m}$. The OrcaFlex model with 120 segments gives a corresponding value of -57.78257m which is a difference from the analytic solution of 0.0072%.

# 3 Natural frequencies of a beam

## 3.1 Introduction

Timoshenko & Gere, Theory of Elastic Stability, 2nd edition, McGraw-Hill, 1961, Section 2.22, pp158-159, considers the stability and transverse vibration of a pin-ended beam. Weight forces are neglected.

Such a beam is easily modelled in OrcaFlex resulting in a close match to theoretical values.

## 3.2 Vibration theory

The natural frequencies for the first mode of vibration are given by:

$$\omega^2 = \frac{g\pi^2}{ql^2}\left(\frac{\pi^2 EI}{l^2} - P\right)$$

where $q/g$ is the mass per unit length, $l$ is beam length, $EI$ is bending stiffness and $P$ is axial compressive load.

### 3.3 Comparison of OrcaFlex with theory

For comparison with OrcaFlex, we take the following arbitrary values:

| | |
|---|---|
| $q/g$ | 1.0 te/m |
| $l$ | 10.0 m |
| $EI$ | 1,000 kN.m$^2$ |
| $P$ | Various values (see below) |

**Table 3: Data for beam**

We set the line diameter such that the line was exactly neutrally buoyant which effectively means that the weight forces are neglected, as in the theory. Four cases have been analysed for two levels of segmentation. Results are reported from the OrcaFlex modal analysis, and from a time domain analysis in which the beam is given a small initial deflection at mid-length.

Natural periods for the first mode of vibration are given in the table below:

| $P$ (kN) | Natural period from theory (s) | Number of segments in OrcaFlex model | Natural period from modal analysis (s) | Natural period from time domain (s) |
|---|---|---|---|---|
| 0 | 2.013 | 10 | 2.030 | 2.031 |
| | | 50 | 2.014 | 2.014 |
| 10 | 2.124 | 10 | 2.142 | 2.141 |
| | | 50 | 2.124 | 2.124 |
| 50 | 2.866 | 10 | 2.902 | 2.902 |
| | | 50 | 2.867 | 2.867 |
| 90 | 6.782 | 10 | 7.150 | 7.130 |
| | | 50 | 6.796 | 6.778 |

**Table 4: Comparison of beam natural periods**

Both the OrcaFlex modal analysis and time domain results show excellent agreement for the range of P considered. As would be expected we see closer agreement for the models with more segmentation.

# 4 Cantilever beam

### 4.1 Introduction

Formulae for deflection, moment, and slope of a cantilever beam are well known. For example see Roark's Formula's for Stress and Strain, 7th edition, McGraw-Hill, 2002, Table 8.1-2a, p191. Cantilever beams are easily modelled in OrcaFlex and we achieve excellent agreement with theoretical results.

### 4.2 Cantilever beam theory

We assume that the beam is horizontal, encastré at one end and free at the other end. We assume a uniform vertical load, due to the beam's self weight. The beam can be specified in terms of weight per unit length, $w_a$, length $l$ and bend stiffness $EI$.

Standard beam theory gives the following values:

| | |
|---|---|
| Deflection at free end | $-\dfrac{w_a l^4}{8EI}$ |
| Moment at fixed end | $\dfrac{-w_a l^2}{2}$ |
| Slope at free end | $\dfrac{w_a l^3}{6EI}$ |

**Table 5: Cantilever beam theory**

### 4.3 Comparison of OrcaFlex with theory

For comparison with OrcaFlex, we take the following arbitrary values:

| | |
|---|---|
| $w_a$ | 1.667 kN/m |
| $l$ | 13.0 m |
| $EI$ | 25,000 kN.m$^2$ |

**Table 6: Data for cantilever**

The theory assumes that the beam is inextensible, and so we used a large value of 1e9kN for $EA$.

For this case we compared the theoretical values with OrcaFlex using models with 10, 20 and 50 segments. The results are tabulated below:

| Theoretical results | | Number of segments in OrcaFlex model | Results from OrcaFlex | |
|---|---|---|---|---|
| Deflection (m) | Slope (°) | | Deflection (m) | Slope (°) |
| -0.23806 | 1.3989 | 10 | -0.24037 | 1.4056 |
| | | 20 | -0.23859 | 1.4004 |
| | | 50 | -0.23809 | 1.3989 |

**Table 7: Comparison of cantilever theory with OrcaFlex**

From the above table it is clear that even the 10 segment model gives very good agreement with the theoretical values. As the number of segments is increased then the difference from theory reduces.

### 4.4 Discussion of moment results

We have not included the results for moment in the above comparison because they require slightly different treatment. The cantilever beam theory for moment neglects the effect of deflection – in effect it assumes that the beam is horizontal.

OrcaFlex's calculation fully accounts for the deflection. Because of this we would expect to see a difference between OrcaFlex's reported fixed end moment and the value predicted by theory. The size of this difference will depend on how much deflection is present. Cases with smaller deflections will have smaller differences.

To test this we varied the deflection by means of varying the bend stiffness value $EI$. For sake of simplicity we did not vary the discretisation of the OrcaFlex model for the moment comparisons – we used 50 segments throughout.

| Theory (kN.m) | EI (kN.m$^2$) | OrcaFlex (kN.m) |
|---|---|---|
| -140.861500 | $25{\times}10^2$ | -139.044489 |
| | $25{\times}10^3$ | -140.842599 |
| | $25{\times}10^4$ | -140.861310 |
| | $25{\times}10^5$ | -140.861498 |

**Table 8: Comparison of cantilever end moments**

As expected, the difference between theory and the OrcaFlex result decreases as the cantilever deflection is decreased.